AD-A116 165

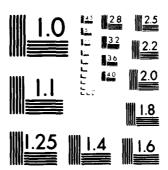
WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER
AN ANALYTICAL, ONE-PARAMETER FAMILY OF SELF-ADJOINT BOUNDARY CO-ETC(U)
APR 82 R L SACHS
UNCLASSIFIED

MRC-TSR-2357

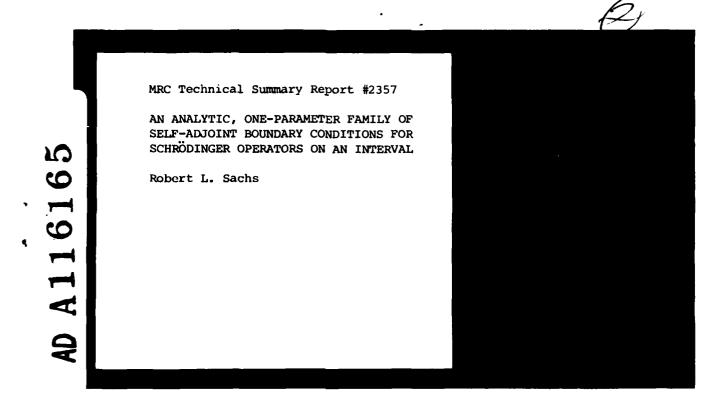
APR 62 R L SACHS

DAG29-80-C-0041
NL

END
7-82
porc



MICROCOPY RESOLUTION TEST CHART NATIONAL BUREAU OF STANDARDS 1965 A



Mathematics Research Center University of Wisconsin—Madison 610 Walnut Street Madison, Wisconsin 53706

April 1982

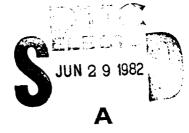
Received February 9, 1982

FILE COP

\ \ \ \

Sponsored by

U. S. Army Research Office P. O. Box 12211 Research Triangle Park North Carolina 27709



Approved for public release
Distribution unlimited

National Science Foundation Washington, D. C. 20550

82 06 29 029

UNIVERSITY OF WISCONSIN - MADISON MATHEMATICS RESEARCH CENTER

AN ANALYTIC, ONE-PARAMETER FAMILY OF SELF-ADJOINT BOUNDARY CONDITIONS FOR SCHRÖDINGER OPERATORS ON AN INTERVAL

Robert L. Sachs

Technical Summary Report #2357 April 1982

ABSTRACT

A one-parameter family of real, homogeneous boundary conditions on the interval [0,1], under which the operator $-\frac{d^2}{dx^2}$ is self-adjoint, is constructed. The relation between such boundary conditions and Lagrangian planes in \mathbf{R}^4 is used and the resulting circle of boundary conditions is seen to include Dirichlet, Neumann, periodic, antiperiodic, and several other well-known examples.

AMS(MOS) Subject Classifications: 34B10; 34B25

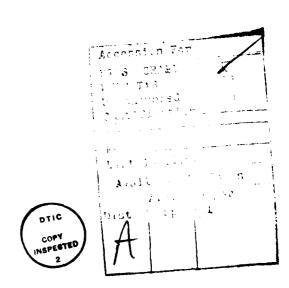
Key Words: Deformation of Boundary Conditions; Self-adjointness.

Work Unit #1 - Applied Analysis

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041. This material is based upon work supported by the National Science Foundation under Grant No. MCS-7927062, Mod. 1.

SIGNIFICANCE AND EXPLANATION

A more or less explicit deformation of boundary conditions for an interval is constructed, under which the operator $-\frac{\mathrm{d}^2}{\mathrm{dx}^2}$ remains self-adjoint. The deformation depends analytically on its parameter and includes Dirichlet, Neumann, periodic, antiperiodic, and other boundary conditions. It is hoped that this family of self-adjoint boundary conditions can be used to construct solutions in problems where one set of boundary conditions (for example, periodic or Dirichlet) leads to a significant simplification of the problem.



The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

AN ANALYTIC, ONE-PARAMETER FAMILY OF SELF-ADJOINT BOUNDARY CONDITIONS FOR SCHRÖDINGER OPERATORS ON AN INTERVAL

Robert L. Sachs

Consider the operator $L = -\frac{d^2}{dx^2}$ acting on reasonably nice functions which satisfy the pair of real linear homogenous boundary conditions

(1)
$$a_i y(0) + b_i y'(0) + c_i y(1) + d_i y'(1) = 0$$
, $i = 1,2$.

By definition, this operator is self-adjoint if and only if the bilinear form:

(2)
$$B(y,z) \equiv y(0) z'(0) - y'(0)z(0) - y(1)z'(1) + y'(1)z(1)$$

vanishes identically for all u,v satisfying the boundary conditions (1). In terms of column vectors $Y \equiv (y(0),y'(1),y'(0),y(1))^T$, $Z \equiv (z(0),z'(1),z'(0),z(1))^T$, (2) is equivalent to

(3)
$$Y^T J Z = 0$$
 where J is the usual 4×4 symplectic matrix

 $\begin{pmatrix} 0 & I \\ -T & 0 \end{pmatrix}$ where 0,I are 2 × 2 matrices. If Y,Z are in the span of

$$\begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \\ \delta_1 \end{pmatrix}, \begin{pmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \\ \delta_2 \end{pmatrix}$$
 then (3) is equivalent to $(A^T B^T)J(A^B) = 0$ where

 $A \equiv \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix}, \quad B \equiv \begin{pmatrix} \gamma_1 & \gamma_2 \\ \delta_1 & \delta_2 \end{pmatrix} \quad \text{and this leads immediately to the condition}$

$$A^{T}B - B^{T}A = 0$$

which is clearly no stronger than the requirement

(5)
$$A + iB$$
 is unitary.

We shall construct a one-parameter family of unitary matrices U(t) connecting the matrix representing periodic boundary conditions:

(6)
$$y(0) = y(1); y'(0) = y'(1)$$

with the matrix representing Dirichlet boundary conditions:

(7)
$$y(0) = y(1) = 0.$$

(6) is equivalent to Y e span
$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$
 which leads to the unitary matrix
$$\begin{bmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ i/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$
 while Dirichlet boundary conditions (7) are

equivalent to the unitary matrix $\begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}$. Thus we seek a one-parameter family U(t) of unitary matrices with

(8)
$$U(0) = \begin{pmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$
, $U(1) = \begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}$.

Such a family is easily found in the form

(9)
$$U(t) = U(0)e^{iSt}$$
 where $e^{iS} = U(0)^{-1}U(1) = \begin{cases} -i\sqrt{2} & i/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{cases}$

i.e.

(10) is = log
$$\begin{cases} -i/\sqrt{2} & i/\sqrt{2} \\ & & = \log(U(0)^{-1}U(1)). \end{cases}$$

$$1/\sqrt{2} & 1/\sqrt{2} & = \log M$$

Now $M = U(0)^{-1} U(1)$ has eigenvalues $\lambda = e^{-\pi i/4} (e^{\pm \pi i/3}) = e^{\pi i/12}$, $e^{-7\pi i/12}$ and, diagonalizing M , we find

(11)
$$M = P \begin{pmatrix} e^{\pi i/12} & 0 \\ 0 & e^{-7\pi i/12} \end{pmatrix} P^{-1} \text{ where } P, P^{-1} \text{ are given by the}$$

$$2 \times 2 \text{ matrices}$$

(12)
$$p = \begin{cases} \frac{\sqrt{3}-1}{\sqrt{2}} e^{\pi i/4} & -(\frac{\sqrt{3}+1}{\sqrt{2}})e^{\pi i/4} \\ 1 & 1 \end{cases}$$

$$\mathbf{p}^{-1} = \begin{pmatrix} \frac{e^{-\pi i/4}}{\sqrt{6}} & \frac{\sqrt{3}+1}{2\sqrt{3}} \\ \frac{-e^{-\pi i/4}}{\sqrt{6}} & \frac{\sqrt{3}-1}{2\sqrt{3}} \end{pmatrix}$$

Thus
$$e^{iSt} = p \begin{pmatrix} e^{\pi i/12t} & 0 \\ & & \\ 0 & e^{-7\pi i/12t} \end{pmatrix}$$

and the desired path is

(13)
$$U(t) = \begin{cases} 1/\sqrt{2} & i/\sqrt{2} \\ & \\ i/\sqrt{2} & 1/\sqrt{2} \end{cases} e^{iSt}$$

which, by a tedious computation, is precisely the matrix

(14)
$$U(t) =$$

$$\frac{\left[\frac{e^{-\pi i/4t}}{\sqrt{2}}\left(\cos(\pi/3t) - \frac{\sin\pi/3t}{\sqrt{3}}\right) - \frac{e^{-\pi i/4t}}{\sqrt{2}}\left(i\cos\frac{\pi}{3}t + (\frac{-2+i}{\sqrt{3}})\sin\frac{\pi}{3}\right)\right]}{\left[\frac{e^{-\pi i/4t}}{\sqrt{2}}\left(i\cos(\pi/3t) + \frac{(2+i)}{\sqrt{3}}\sin(\frac{\pi}{t})\right) - \frac{e^{-\pi i/4t}}{\sqrt{2}}\left(\cos\pi/3t - \frac{1}{\sqrt{3}}\sin\pi/3t\right)\right]}$$

We see easily that U has period 24 and that U(t + 12) = -U(t), indeed U(t+6) = iU(t) so that, in terms of the corresponding boundary conditions, we need only consider U(t), $0 \le t \le 12$. Listing U(j), $j = 0, \cdots, 11$ and the corresponding boundary conditions, we have

$$(15) \begin{cases} u(0) = \begin{pmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}, \quad u(1) = \begin{pmatrix} 0 & +i \\ 1 & 0 \end{pmatrix}$$

$$U(2) = \begin{pmatrix} i/\sqrt{2} & i/\sqrt{2} \\ -i/\sqrt{2} & i/\sqrt{2} \end{pmatrix} , \quad U(3) = \begin{pmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1+i}{2} \end{pmatrix}$$

$$U(4) = \begin{bmatrix} 0 & \frac{-1+i}{\sqrt{2}} \\ \frac{1+i}{\sqrt{2}} & 0 \end{bmatrix}, \quad U(5) = \begin{bmatrix} \frac{-1+i}{2} & \frac{-1+i}{2} \\ \frac{1-i}{2} & \frac{-1+i}{2} \end{bmatrix}$$

$$U(6) = \begin{cases} i/\sqrt{2} & -1/\sqrt{2} \\ & & \\ -1/\sqrt{2} & i/\sqrt{2} \end{cases} \quad i \quad U(7) = \begin{cases} 0 & -1 \\ & & \\ i & 0 \end{cases}$$

$$U(8) = \begin{cases} -1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{cases} ; \quad U(9) = \begin{cases} \frac{-1+i}{2} & \frac{-1-i}{2} \\ \frac{-1-i}{2} & \frac{-1+i}{2} \end{cases}$$

$$U(10) = \begin{bmatrix} 0 & \frac{-1-i}{\sqrt{2}} \\ \frac{-1+i}{\sqrt{2}} & 0 \end{bmatrix} , \quad U(11) = \begin{bmatrix} \frac{-1-i}{2} & \frac{-1-i}{2} \\ \frac{1+i}{2} & \frac{-1-i}{2} \end{bmatrix}$$

The corresponding boundary conditions, found by reversing the procedure above, are as follows:

t	Boundary conditions
0	y(0) = y(1) , $y'(0) = y'(1)$
1	y(0) = 0 , $y(1) = 0$
2	y(0) = 0 , $y'(1) = 0$
3	$y(0) = -y^*(1), y^*(0) = y(1)$
4	y(0) = -y'(0), y'(1) = y(1)
5	$y(0) = -y^*(0), y^*(1) = -y(1)$
6	y(0) = -y(1) , $y'(0) = -y'(1)$
7	$y^*(0) = 0$, $y^*(1) = 0$
8	$y^*(0) = 0$, $y(1) = 0$
9	y(0) = y'(1), $y'(0) = -y(1)$
10	y(0) = y'(0) , $y'(1) = -y(1)$
11	y(0) = y'(0) , $y'(1) = y(1)$

Thus our one parameter family in fact includes periodic, Dirichlet, antiperiodic, Neumann, and several other well-known boundary conditions.

RLS/db

SECURITY CLASSIFICATION OF THIS PAGE (When Date Entered)

REPORT DOCUMENT	READ INSTRUCTIONS BEFORE COMPLETING FORM		
1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER	
2357	AD-A11616	5	
4. TITLE (and Subtitie)	172 7776	5. TYPE OF REPORT & PERIOD COVERED	
AN ANALYTIC, ONE-PARAMETER FA	Summary Report - no specific		
BOUNDARY CONDITIONS FOR SCHRO	reporting period		
AN INTERVAL	6. PERFORMING ORG. REPORT NUMBER		
7. AUTHOR(e)		8. CONTRACT OR GRANT NUMBER(#)	
Robert L. Sachs		DAAG29-80-C-0041	
		MCS-7927062, Mod 1	
9. PERFORMING ORGANIZATION NAME AND		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS	
Mathematics Research Center	1 - Applied Analysis		
610 Walnut Street Wisconsin		1 - Applied Analysis	
Madison, Wisconsin 53706			
11. CONTROLLING OFFICE NAME AND ADDRE	E5S	12. REPORT DATE	
See Item 18		April 1982	
Dec Item 10		5	
14. MONITORING AGENCY NAME & ADDRESS	il different from Controlling Office)	15. SECURITY CLASS. (of thie report)	
		UNCLASSIFIED	
		154. DECLASSIFICATION/DOWNGRADING	
	•	SCHEDULE	
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)			
18. SUPPLEMENTARY NOTES U. S. Army Research Office		ence Foundation	
P. O. Box 12211 Washington, D. C. 20550 Research Triangle Park, North Carolina 27709			
19. KEY WORDS (Continue on reverse side if necessary and identify by block number)			
19. KEY WORDS (Continue on reverse side if nec	coonry and identify by block number)		
19. KEY WORDS (Continue on reverse side if nec Deformation of Boundary Condi			

DD 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

#